Representability of algebras with ascending chain condition on ideals, NCRA VII, Lens, France, 2021, in honor of Tariq Rizvi

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(joint work with B. Greenfeld)

July 6, 2021

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In this talk, based on joint work with Be'eri Greenfeld, we sketch the current situation concerning representability of PI (polynomial identity)-rings satisfying ACC (ascending chain condition) on ideals. We present a non-representable example, and some positive results concerning left Noetherian PI-rings. These require results of independent interest concerning a construction of Lewin-Bergman-Dicks-Anan'in. The general question of the representability of Noetherian PI-rings (even Artinian PI-rings) remains open.

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R is called **representable** if K can be taken to be a field.

Representability of algebras is a venerable subject, of great importance since it enables us to use techniques of matrix theory, such as the trace. It was explored in depth by PI theorists in the 1970s since every representable algebra obviously is PI, satisfying the identities of $M_n(K)$. Obviously, any f.d. algebra over a field is representable.

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Any semiprime PI-algebra satisfying the ACC on annihilator ideals is representable.

But a homomorphic image of a representable algebra need not be representable.

Bergman, *Some examples in PI ring theory*, Israel J. Math. **18** (1974), 1–5, threatened to destroy the subject by showing for any prime number p that the endomorphism ring $\mathrm{End}_{\mathbb{Z}}(\mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/p^2\mathbb{Z})$ is a ring (of p^5 elements) that is not weakly representable.

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Lewin proved that uncountably many affine PI-algebras over $\mathbb{Q}$ are not representable. (The argument is straightforward: There are uncountably many isomorphism classes of affine PI-algebras, whereas only countably many of them are representable.) But they all are homomorphic images of representable PI-algebras.

Small (1971) found an explicit example of an affine PI-algebra over an arbitrary field, that is not representable. In view of Small's example, researchers looked for extra conditions to guarantee representability of affine PI -algebras, many of which were knocked down by Irving, Irving-Small, and L'vov-Markov.

On the other hand, Anan'in proved a number of positive results, culminating in *The representability of finitely generated algebras with chain condition*, Arch. Math. **59** (1992), in which he showed that any left Noetherian affine PI-algebra is representable.

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His approach was to generalize a result of Lewin, *On some infinitely presented associative algebras*, Collection of articles dedicated to the memory of Hanna Neumann, III. J. Austral. Math. Soc. **16** (1973), 290-293 (proved more conceptually by Bergman and Dicks *Universal derivations*, J. Algebra **36** (1975), 193–211), providing the following embedding of any PI-ring R into a generalized upper triangular matrix ring (cf. the lecture earlier today by Prof. Ashraf):

Given any ring R and two homomorphisms $\sigma_i : R \to R_i$ $i = 1, 2$, they seek an optimal solution for the problem:

$$\theta : R \to \tilde{R} := \begin{pmatrix} R_1 & \Omega_R(R_1, R_2) \\ 0 & R_2 \end{pmatrix}$$

where $\Omega_R(R_1, R_2)$ is an $R_1 - R_2$-bimodule and the map is given by:

$$\theta : r \mapsto \begin{pmatrix} \sigma_1(r) & \delta(r) \\ 0 & \sigma_2(r) \end{pmatrix}.$$

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It turns out that for this map to be a ring homomorphism it is necessary and sufficient that δ be a (σ_1, σ_2)-derivation.
Lewin and Bergman-Dicks find the solution with minimal possible kernel, which turns out to be P_1P_2 (where $P_i = \ker \sigma_i$) when R is an algebra over a field.

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Anan'in generalized this result to an arbitrary set of ideals $P_1, \dots, P_m$.

A non-representable affine PI-algebra over an arbitrary field, satisfying ACC on ideals

In contrast to Anan'in's theorem, here is a surprisingly straightforward example by Greenfeld, ArXiv 2008.11041v.2, of a non-weakly representable affine PI-algebra satisfying ACC on ideals. Moreover, the quotient modulo its nilpotent radical N is a polynomial ring in one variable, the significance of which will be seen.

Let W be an F-algebra and M a W-bimodule. Given an F-linear map $B : M \otimes_W M \to F$, we can define an F-algebra:

$$A = \begin{pmatrix} F & M & F \\ 0 & W & M \\ 0 & 0 & F \end{pmatrix}$$

whose multiplication is given by:

$$\begin{pmatrix} \alpha_1 & v_1 & \lambda \\ 0 & w & v_2 \\ 0 & 0 & \alpha_2 \end{pmatrix} \begin{pmatrix} \alpha_1' & v_1' & \lambda' \\ 0 & w' & v_2' \\ 0 & 0 & \alpha_2' \end{pmatrix}$$
$$= \begin{pmatrix} \alpha_1\alpha_1' & \alpha_1 v_1' + v_1 w' & \alpha_1\lambda' + \alpha_2'\lambda + B(v_1, v_2') \\ 0 & ww' & wv_2' + \alpha_2' v \\ 0 & 0 & \alpha_2\alpha_2' \end{pmatrix}.$$

Since B is defined over $M \otimes_W M$, this multiplication law endows A with a well-defined F-algebra structure. The only nontrivial verification needed to show associativity involves the $1,2$ and $2,3$ positions.

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If W is an affine integral domain, the algebra A is an affine PI-algebra with $N^3 = 0$ and $R/N \cong F \times W \times F$.

We now take $W = F[t]$ and let $M = Fu_1 + Fu_2 + \cdots$ be a countable dimensional F-vector space. We consider M as an $F[t]$-bimodule through:

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$$B(u_i, u_j) = \begin{cases} 1, & \text{if} \quad \exists t \geq 1 : \ i + j = 2^t \\ 0, & \text{otherwise} \end{cases}$$

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B is a well-defined F-linear map defined over $M \otimes_{F[t]} M$ and thus $A = \begin{pmatrix} F & M & F \\ 0 & F[t] & M \\ 0 & 0 & F \end{pmatrix}$ is a well defined F-algebra.

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The algebra A satisfies ACC on ideals.

Proof (sketch): Any ideal $I \triangleleft A$ is of finite codimension inside a subspace of the form:

$$\begin{pmatrix} E & V_1 & L \\ 0 & J & V_2 \\ 0 & 0 & K \end{pmatrix}$$

where $E, K, L \in \{0, F\}$, $J \in \{0, F[t]\}$ and $V_i \in \{0, M\}$. It follows that any ascending chain of ideals of A stabilizes.

The role of irreducible algebras

An algebra is **irreducible** if the intersection of any two nonzero ideals is nonzero.

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A key observation: Any algebra with ACC on ideals is a finite subdirect product of irreducible algebras, so the question of its representability reduces to this question for irreducible algebras.

Left Noetherian algebras finite over their center

Ananin's theorem on the representability of left Noetherian algebras relies on presenting the finitely many generators inside matrices. When one drops the hypothesis of "affine," the situation becomes more opaque, but some results are available.

When an irreducible left Noetherian algebra A is finite as a module over its center, R. and Small, *Representability of algebras finite over their centers*, Journal of Algebra 442, 506–524 (2015), used a theorem of Wehrfritz to obtain a “coefficient subfield” over which A is representable.

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Hence any left Noetherian algebra A finite over a commutative (not necessarily central) subalgebra is representable.

Left Noetherian algebras containing a distinguished subalgebra

In the counterexamples satisfying ACC on ideals, A/N is isomorphic to a subalgebra of A, a condition that is related to cohomology. This makes the following result of Greenfeld-R. of interest:

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Let R be a left Noetherian algebra over a field, containing a subalgebra $W \subseteq R$ satisfying ACC on ideals, such that R/N is a finitely generated left module over $\overline{W}$, the reduction of W modulo N, satisfying the condition:

$W[c^{-1}]$ is finite over its center, for some c of $C := \mathrm{Cent}(W)$ which is regular in R.

Then R is representable.

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Then R is representable.

For example, one could take $W = R/N$, the case in the examples presented here.

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Amitsur, Small, and Rowen proved that if a PI-ring R has two maximal ideals whose product is 0 then R is weakly-representable.

Their method is to reduce the Lewin-Bergman-Dicks construction to the case where the diagonal components are central. This only works when there are two components, leading one to see whether one can generalize to three or more components.

A semiprimary PI-algebra over an arbitrary field that is not weakly representable

A counterexample for semiprimary PI-algebras is obtained by extending the previous construction. We take $A = F(t)$ and $M = V$ a 1-dimensional $F(t)$-vector space, which we naturally identify with $F(t)$. Then $V \otimes_{F(t)} V \cong V$ is an F-vector space via $v \otimes w \mapsto vw$. We fix an F-linear basis for $F(t)$, say, $\mathfrak{B}$ containing $1, t, t^2, \dots$ and define $\widetilde{B} : V \otimes_{F(t)} V \to F$ on basis elements as follows:

$$\widetilde{B}(1, v) = \begin{cases} 1, & \text{if} \quad \exists k \geq 1 : \ v = t^{2^k} \\ 0, & \quad \text{otherwise} \end{cases}$$

We can therefore form, in the same manner as before, the semiprimary F-algebra:

$$S = \begin{pmatrix} F & V & F \\ 0 & F(t) & V \\ 0 & 0 & F \end{pmatrix}$$

Since it contains the previous example, it is not weakly representable.

Remaining open questions

1. (In light of Bergman's example) Is every PI-ring embeddible into an endomorphism ring over a commutative ring?

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3. If (2) holds, is every left Noetherian PI-algebra R representable? One might expect the construction $\tilde{R}$ of Lewin-Bergman-Dicks to be relevant since $\tilde{R}/\tilde{N}$ embeds into $\tilde{R}$, but annoyingly $\tilde{R}$ is no longer left Noetherian.

Thank you, congratulations to André for your immense effort in organizing this conference, and happy birthday to my friend Tariq, a person of integrity and honor.